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A perturbation result for q-open quotient morphisms in normed spaces and applications to linear relations

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ABSTRACT

Using some techniques of perturbation theory for quotient morphisms between Banach spaces we obtain necessary and sufficient conditions for the stability of the topological index of a q-open quotient morphism under small (with respect to the gap topology) perturbations with quotient morphisms. As an application we obtain necessary and sufficient conditions for the stability of the topological index of an open linear relation with closed multivalued part between normed spaces under small perturbations with linear relations.

1. Introduction

In [2] (see also [1,7]) the authors have obtained the stability of the index of a quotient morphism between Banach spaces under small perturbations (with respect to the gap topology [14]). Their result generalizes some pioneering work due to Kato [14, Chapter IV, Sections 4 and 5]. For more on the theory of quotient morphisms we refer to [15–17]. The aim of this paper is to study (using Albrecht–Vasilescu result, see Theorem 3) the behavior of the topological index of a quotient morphism between normed spaces under small perturbations (Theorem 1).

As an application we obtain necessary and sufficient conditions for the stability of the topological index of an open linear relation (multivalued operator) with closed multivalued part between normed spaces under small perturbations with linear relations (Theorem 2). In connection with our results on linear relations we refer to [3–5,10].

The paper is organized as follows. In Section 1 we introduce some notation, definitions and state our main result and its application to linear relations. In Section 2 we describe our main tool as well as some duality results which will be very useful in the sequel. In Sections 3 and 4 we prove the results stated in Section 1 and obtain several interesting consequences.

1.1. Basic definitions and notation

Let X, Y be two normed spaces and X', Y' be their topological dual. We denote by $\mathcal{B}(X,Y)$ the normed space of continuous linear operators acting from X into Y. If $T \in \mathcal{B}(X,Y)$ then $T' \in \mathcal{B}(Y',X')$ denotes the adjoint of T. The operator T is called *open* if whenever U is a neighbourhood in X, the image $T(U)$ is a neighbourhood in $R(T)$. Note that T is open iff there exists $\rho > 0$ such that $\rho B_Y \cap R(T) \subset T(B_X)$, where B_X and B_Y denote the closed unit balls of X and Y respectively, or equivalently iff there exists $k > 0$ such that for any $y \in R(T)$ there exists $x \in X$ such that $Tx = y$ and $\|x\| \leqslant k\|y\|$.

Consider now $\mathcal{X}$ and $\mathcal{Y}$ two normed spaces. A *quotient morphism* (or simply a q-morphism) of $\mathcal{X}$ into $\mathcal{Y}$ is a linear mapping $T : X/X_0 \to Y/Y_0$ where $X_0 \subset X \subset \mathcal{X}$ and $Y_0 \subset Y \subset \mathcal{Y}$ are linear subspaces of $\mathcal{X}$ and $\mathcal{Y}$ respectively. The *range* of the q-morphism T is denoted by $R(T)$ and the *kernel* of T is denoted by $N(T)$. In [1] the authors have introduced two important notions associated to T, namely the *lifted graph* of T denoted $G_0(T)$ and defined by

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$$G_0(T) := \{(x, y) \in X \times Y;\ T(x + X_0) = y + Y_0\}$$

and the *lifted range* of T denoted $R_0(T)$ and defined by

$$R_0(T) := \{y \in Y;\ \exists x \in X\colon\ T(x + X_0) = y + Y_0\}.$$

Throughout this paper, for a q-morphism $T : X/X_0 \to Y/Y_0$, the subspace Y_0 will be considered closed in $\mathcal{Y}$. Therefore the space Y/Y_0 will be a normed space. On the other hand the closure of X_0 will be taken in $\mathcal{X}$, the closure of Y will be taken in $\mathcal{Y}$ and the closure of $G_0(T)$ will be taken in $\mathcal{X} \times \mathcal{Y}$. Finally, the closure of $R(T)$ will be taken in Y/Y_0.

If T is a q-morphism which satisfies

$$\dim(N(T)) < \infty \quad \text{or} \quad \operatorname{codim}(\overline{R(T)}) < \infty,$$

then we define the *topological index* of T by

$$\overline{\operatorname{ind}}(T) = \dim(N(T)) - \operatorname{codim}(\overline{R(T)}).$$

We say that the q-morphism $T : X/X_0 \to Y/Y_0$ is *q-open* if the linear operator

$$\phi_T : X \to Y/Y_0, \quad \phi_T(x) = T(x + X_0)$$

is open as an operator from the normed space X into the normed space Y/Y_0. According to [7, Definition I.6.6] (see also [14]), if M and N are closed subspaces of a normed space X, then the *gap* between M and N is defined by

$$\widehat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\},$$

where

$$\delta(M, N) = \sup_{x \in M,\ \|x\| \leqslant 1} \inf_{y \in N} \|x - y\|.$$

1.2. Main results

Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X, $\widetilde{X}$, X_0 be subspaces of $\mathcal{X}$ such that $X_0 \subset X \cap \widetilde{X}$, Y, $\widetilde{Y}$ be subspaces of $\mathcal{Y}$ and Y_0, $\widetilde{Y}_0$ be closed subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$, $\widetilde{Y}_0 \subset \widetilde{Y}$. Our main result goes as follows.

Theorem 1. *Let $T : X/X_0 \to Y/Y_0$ be q-morphism which is q-open and satisfies $\overline{X}_0 \times Y_0 \subset G_0(T)$,*

$$\dim(N(T)) < \infty \quad \textit{and} \quad \operatorname{codim}(\overline{R(T)}) < \infty.$$

Then there exists $\epsilon_T > 0$ such that for every q-morphism $\widetilde{T} : \widetilde{X}/X_0 \to \widetilde{Y}/\widetilde{Y}_0$ with $\overline{X}_0 \times \widetilde{Y}_0 \subset G_0(\widetilde{T})$,

$$\widehat{\delta}(\overline{G_0(T)}, \overline{G_0(\widetilde{T})}) < \epsilon_T, \qquad \widehat{\delta}(Y_0, \widetilde{Y}_0) < \epsilon_T \quad \textit{and} \quad \widehat{\delta}(\overline{Y}, \overline{\widetilde{Y}}) < \epsilon_T$$

one has that

$$\dim(N(\widetilde{T})) \leqslant \dim(N(T)), \qquad \operatorname{codim}(\overline{R(\widetilde{T})}) \leqslant \operatorname{codim}(\overline{R(T)})$$

and

$$\overline{\operatorname{ind}}(\widetilde{T}) \leqslant \overline{\operatorname{ind}}(T).$$

Moreover, $\overline{\operatorname{ind}}(\widetilde{T}) = \overline{\operatorname{ind}}(T)$ *iff $\widetilde{T}$ is q-open.*

Note that in the preceding theorem the spaces X_0, Y and $\widetilde{Y}$ are not necessarily closed and no continuity is required upon T and $\widetilde{T}$. So, even in the Banach space context, our main result completes our main tool due to Albrecht and Vasilescu. On the other hand, Theorem 1 is well adapted for applications to linear relations in normed spaces (see Theorem 2 bellow).

Let X, Y be normed spaces and Z be a linear subspace of $X \times Y$. Following Arens [8], we say that Z is a *linear relation* between X and Y. If Z is a linear relation, then we associate it the following linear subspaces: $D(Z) = \{x \in X;\ \exists y \in Y\colon\ (x, y) \in Z\}$ called the domain of Z, $R(Z) = \{y \in Y;\ \exists x \in X\colon\ (x, y) \in Z\}$ called the range of Z, $N(Z) = \{x \in D(Z);\ (x, 0) \in Z\}$ called the kernel of Z and $M(Z) = \{y \in R(Z);\ (0, y) \in Z\}$ its multivalued part. For $x \in D(Z)$, the set $\{y \in Y;\ (x, y) \in Z\}$ is denoted by $Z(x)$. A linear relation Z can also be identified with the graph of the *multivalued linear operator* $T : D(Z) \to \mathcal{P}(Y) \setminus \{\emptyset\}$ defined by $Tx = Z(x)$ $(x \in D(Z))$, where $\mathcal{P}(Y)$ denotes the family of all subsets of Y.

A linear relation Z is said to be *open* if whenever U is a neighbourhood in $D(Z)$, the image $Z(U) := \bigcup_{x \in U} Z(x)$ is a neighbourhood in $R(Z)$. Note that Z is open iff there exists $\rho > 0$ such that

$$\rho B_Y \cap R(Z) \subset Z(B_X \cap D(Z))$$

(see [10, Propositions II.2.4 and II.3.2(b)]).

Assume that the linear relation Z satisfies

$$\dim\bigl(N(Z)\bigr) < \infty \quad \text{or} \quad \operatorname{codim}\bigl(\overline{R(Z)}\bigr) < \infty.$$

Then, the quantity

$$\overline{\operatorname{ind}}(Z) = \dim\bigl(N(Z)\bigr) - \operatorname{codim}\bigl(\overline{R(Z)}\bigr)$$

is called the *topological index* of Z (see [10,13]).

Theorem 2. *Let X, Y be normed spaces and $Z_1 \subset X \times Y$ be an open, linear relation such that $M(Z_1)$ is closed,*

$$\dim\bigl(N(Z_1)\bigr) < \infty \quad \textit{and} \quad \operatorname{codim}\bigl(\overline{R(Z_1)}\bigr) < \infty.$$

There exists $\epsilon_{Z_1} > 0$ such that for any linear relation $Z_2 \subset X \times Y$ with $M(Z_2)$ closed,

$$\widehat{\delta}(\overline{Z_1}, \overline{Z_2}) < \epsilon_{Z_1} \quad \textit{and} \quad \widehat{\delta}\bigl(M(Z_1), M(Z_2)\bigr) < \epsilon_{Z_1},$$

one has that

$$\dim\bigl(N(Z_2)\bigr) \leqslant \dim\bigl(N(Z_1)\bigr), \qquad \operatorname{codim}\bigl(\overline{R(Z_2)}\bigr) \leqslant \operatorname{codim}\bigl(\overline{R(Z_1)}\bigr)$$

and

$$\overline{\operatorname{ind}}(Z_2) \leqslant \overline{\operatorname{ind}}(Z_1).$$

Moreover, $\overline{\operatorname{ind}}(Z_2) = \overline{\operatorname{ind}}(Z_1)$ iff Z_2 is open.

If the linear relations Z_1 and Z_2 have the same multivalued part, then we have shown in [12] that the preceding theorem still holds true even if their multivalued parts are not closed.

For interesting applications of linear relations to differential equations see for example [9,11].

2. Auxiliary results

2.1. Main tool

Let $\mathcal{X}$ be a Banach space and $X_0^i \subset X^i \subset \mathcal{X}$ $(i = 1, 2)$ be closed subspaces of $\mathcal{X}$. A q-morphism $T \in \mathcal{B}(X^1/X_0^1, X^2/X_0^2)$ is called *Fredholm* if

$$\dim\bigl(N(T)\bigr) < \infty \quad \text{and} \quad \operatorname{codim}\bigl(R(T)\bigr) < \infty.$$

In this case, the quantity

$$\operatorname{ind}(T) = \dim\bigl(N(T)\bigr) - \operatorname{codim}\bigl(R(T)\bigr)$$

is called the (algebraic) *index* of T. Note that since $R(T)$ is finite codimensional from [7, Proposition I.7.6] we infer that $R(T)$ is closed. Therefore, in the Banach space context, the topological and the algebraical index of T are the same. The following theorem is our main tool and is due to Albrecht and Vasilescu ([2, Proposition 2.20], [7, Theorem II.1.15]).

Theorem 3. *Let $\mathcal{X}$ be a Banach space and $X_0^i \subset X^i \subset \mathcal{X}$, $\widetilde{X}_0^i \subset \widetilde{X}^i \subset \mathcal{X}$ $(i = 1, 2)$ be closed subspaces of $\mathcal{X}$. Assume that $T \in \mathcal{B}(X^1/X_0^1, X^2/X_0^2)$ is a Fredholm q-morphism. There exists $\epsilon_T > 0$ such that every q-morphism $\widetilde{T} \in \mathcal{B}(\widetilde{X}^1/\widetilde{X}_0^1, \widetilde{X}^2/\widetilde{X}_0^2)$ which satisfies*

$$\widehat{\delta}\bigl(X_0^1, \widetilde{X}_0^1\bigr) < \epsilon_T, \qquad \widehat{\delta}\bigl(X^2, \widetilde{X}^2\bigr) < \epsilon_T \quad \textit{and} \quad \widehat{\delta}\bigl(G_0(T), G_0(\widetilde{T})\bigr) < \epsilon_T$$

is Fredholm and

$$\operatorname{ind}(T) = \operatorname{ind}(\widetilde{T}).$$

Moreover,

$$\dim\bigl(N(\widetilde{T})\bigr) \leqslant \dim\bigl(N(T)\bigr) \quad \textit{and} \quad \operatorname{codim}\bigl(R(\widetilde{T})\bigr) \leqslant \operatorname{codim}\bigl(R(T)\bigr).$$

2.2. Some duality results

In the next lemma we introduce some useful isometric isomorphisms.

Lemma 1. *Let X be a normed space and M, N be linear subspaces of X such that N is closed in X and $N \subset M$. Then, we have the following*:

(i) *The mapping*

$$\varphi : X'/M^{\perp} \to M', \quad \varphi\big(x' + M^{\perp}\big) = x'|M$$

is an isometric isomorphism. We recall that $M^{\perp} := \{x' \in X';\ x'|M = 0\}$.

(ii) *If $\pi : M \to M/N$ is the canonical surjection, then the mapping*

$$\psi : (M/N)' \to N^{\perp}/M^{\perp}, \quad \psi = \varphi^{-1} \circ \pi'$$

is an isometric isomorphism.

(iii) *If in addition M is closed in X, then $\delta(M, N) = \delta(N^{\perp}, M^{\perp})$.*

Proof. For the proof of (i) see [10, Proposition III.1.8(a)] and for the proof of (ii) in the case $M = X$ see [10, Proposition III.1.8(b)]. For the proofs of (ii) and (iii) in the Banach space context see [7, Propositions I.8.5, I.8.10]. □

Remark 1. Let X be a normed space and M, N be subspaces of X such that N is closed in X and $N \subset M$. From the above lemma it follows that

$$\dim(M/N) = \dim\big((M/N)'\big) = \dim\big(N^{\perp}/M^{\perp}\big).$$

In particular, if N is a linear subspace of X, then $\dim(X/\overline{N}) = \dim(N^{\perp})$.

The following lemma gives some duality results and is essential in the proof of the main result of this paper.

Lemma 2. *Let X, Y be two normed spaces, $T \in \mathcal{B}(X, Y)$ and $T' \in \mathcal{B}(Y', X')$ its adjoint. Then, we have the following*:

(i) $\overline{R(T')} \subset N(T)^{\perp}$ *and* $R(T)^{\perp} = N(T')$.
(ii) *The operator T is open iff $R(T') = N(T)^{\perp}$.*
(iii) $\operatorname{codim}(\overline{R(T)}) = \dim(N(T'))$ *and* $\dim(N(T)) \leqslant \operatorname{codim}(\overline{R(T')})$.
(iv) *If T is open then $R(T')$ is closed and* $\dim(N(T)) = \operatorname{codim}(R(T'))$. *Moreover, if $R(T')$ is closed and* $\dim(N(T)) = \operatorname{codim}(R(T')) < \infty$, *then T is open.*

Proof. For the proofs of (i) and (ii) see for example [10, Propositions III.1.4(a), III.4.6(b)] (in the more general context of linear relations). Claims (iii) and (iv) follow easily from (i) and (ii) (see [12] for a complete proof). □

Remark 2. Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X_0, X be subspaces of $\mathcal{X}$ such that $X_0 \subset X$ and X_0 closed in $\mathcal{X}$, Y_0, Y be subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$ and Y_0 closed in $\mathcal{Y}$. Consider $S \in \mathcal{B}(X/X_0, Y/Y_0)$ and $S' \in \mathcal{B}((Y/Y_0)', (X/X_0)')$ its adjoint. Let $\psi_1 : Y_0^{\perp}/Y^{\perp} \to (Y/Y_0)'$ be the isometric isomorphism given in Lemma 1(ii), that is $\psi_1(y_0' + Y^{\perp}) = \eta$, where $\eta(y + Y_0) = y_0'(y)$ for all $y \in Y$. Using the same result, consider also the isometric isomorphism $\psi_2 : (X/X_0)' \to X_0^{\perp}/X^{\perp}$ given by $\psi_2(\xi) = x_0' + X^{\perp}$ where $x_0' \in \mathcal{X}'$ is arbitrary chosen with the property $x_0'(x) = \xi(x + X_0)$ for all $x \in X$. We associated to S' the continuous q-morphism $S^* \in \mathcal{B}(Y_0^{\perp}/Y^{\perp}, X_0^{\perp}/X^{\perp})$ defined by

$$S^* : Y_0^{\perp}/Y^{\perp} \to X_0^{\perp}/X^{\perp}, \quad S^* = \psi_2 S' \psi_1.$$

Let $y_0' \in Y_0^{\perp}$ and $x_0' \in \mathcal{X}'$. Note that $S^*(y_0' + Y^{\perp}) = x_0' + X^{\perp}$ iff for any $x \in X$ one has that $x_0'(x) = y_0'(y)$ where $y \in Y$ is arbitrary chosen with the property $(x, y) \in G_0(S)$.

Lemma 3. *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X_0, X be subspaces of $\mathcal{X}$ such that $X_0 \subset X$ and X_0 closed in $\mathcal{X}$, Y_0, Y be subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$ and Y_0 closed in $\mathcal{Y}$. Consider $S \in \mathcal{B}(X/X_0, Y/Y_0)$ and $S^* \in \mathcal{B}(Y_0^{\perp}/Y^{\perp}, X_0^{\perp}/X^{\perp})$ defined in Remark 2.*

(i) *We have that $G_0(S^*) = J(G_0(S))^{\perp}$, where $J : \mathcal{X} \times \mathcal{Y} \to \mathcal{Y} \times \mathcal{X}$ is the isometric isomorphism given by $J(x, y) = (-y, x)$ for all $(x, y) \in \mathcal{X} \times \mathcal{Y}$.*
(ii) $\operatorname{codim}(\overline{R(S)}) = \dim(N(S^*))$ *and* $\dim(N(S)) \leqslant \operatorname{codim}(\overline{R(S^*)})$.
(iii) *If S is open then $R(S^*)$ is closed in $X_0^{\perp}/X^{\perp}$ and* $\dim(N(S)) = \operatorname{codim}(R(S^*))$. *Moreover, if $R(S^*)$ is closed in $X_0^{\perp}/X^{\perp}$ and* $\dim(N(S)) = \operatorname{codim}(R(S^*)) < \infty$, *then S is open.*

Proof. (i) Let $(y_0', x_0') \in G_0(S^*)$. Using Remark 2 it follows that $x_0'(x) = y_0'(y)$ for any $(x, y) \in G_0(S)$. This implies that $(y_0', x_0')(-y, x) = 0$ for any $(x, y) \in G_0(S)$, that is $(y_0', x_0') \in J(G_0(S))^{\perp}$.

Reciprocally, assume that $(y_0', x_0') \in \mathcal{Y}' \times \mathcal{X}'$ is such that $(y_0', x_0')(-y, x) = 0$ for any $(x, y) \in G_0(S)$. As we clearly have that $\{0\} \times Y_0 \subset G_0(S)$, we deduce that $y_0' \in Y_0^{\perp}$. Using that for any $x \in X$ we have that $x_0'(x) = y_0'(y)$ where $y \in Y$ is arbitrary chosen with the property $(x, y) \in G_0(S)$ and Remark 2, it follows that $S^*(y_0' + Y^{\perp}) = x_0' + X^{\perp}$, that is $(y_0', x_0') \in G_0(S^*)$.

(ii) Let $S' \in \mathcal{B}((Y/Y_0)', (X/X_0)')$ be the adjoint of S. From Lemma 2(iii) one has that $\operatorname{codim}(\overline{R(S)}) = \dim(N(S'))$. On the other hand a simple calculation shows that $N(S') = \psi_1(N(S^*))$. Because ψ_1 is an isomorphism, it follows that $\dim(N(S')) = \dim(N(S^*))$, which together with the equality above imply that $\operatorname{codim}(\overline{R(S)}) = \dim(N(S^*))$. Note that $R(S^*) = \psi_2(R(S'))$ and from Lemma 2(iii), we have that $\dim(N(S)) \leqslant \operatorname{codim}(\overline{R(S')})$. Using that $\psi_2^{-1} : X_0^{\perp}/X^{\perp} \to (X/X_0)'$ is an isometric isomorphism it follows that

$$\overline{R(S')} = \overline{\psi_2^{-1}\big(R(S^*)\big)} = \psi_2^{-1}\big(\overline{R(S^*)}\big)$$

and

$$\dim\big(\big(X_0^{\perp}/X^{\perp}\big)/\overline{R(S^*)}\big) = \dim\big(\big((X/X_0)'\big)/\big(\psi_2^{-1}\big(\overline{R(S^*)}\big)\big)\big).$$

Consequently $\dim(N(S)) \leqslant \operatorname{codim}(\overline{R(S')}) = \operatorname{codim}(\overline{R(S^*)})$.

(iii) Assume now that S is open. Then, from Lemma 2(iv) it follows that $R(S')$ is closed in $(X/X_0)'$ and $\dim(N(S)) = \operatorname{codim}(R(S'))$. Because ψ_2 is an isometric isomorphism it follows that $R(S^*) = \psi_2(R(S'))$ is closed in $X_0^{\perp}/X^{\perp}$ and $\dim(N(S)) = \operatorname{codim}(R(S')) = \operatorname{codim}(R(S^*))$. Assume now that $R(S^*)$ is closed in $X_0^{\perp}/X^{\perp}$ and $\dim(N(S)) = \operatorname{codim}(R(S^*)) < \infty$. Because ψ_2^{-1} is an isometric isomorphism and $\psi_2^{-1}(R(S^*)) = R(S')$ it follows that $R(S')$ is closed in $(X/X_0)'$ and $\operatorname{codim}(R(S^*)) = \operatorname{codim}(R(S'))$. Hence, $\dim(N(S)) = \operatorname{codim}(R(S')) < \infty$ and the openness of S follows now from Lemma 2(iv). $\square$

The corresponding result in the Banach spaces context of the following lemma has been proved in [7, Proposition I.8.15(iii)].

Lemma 4. *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X, $\widetilde{X}$, X_0, $\widetilde{X}_0$ be subspaces of $\mathcal{X}$ such that $X_0 \subset X$, $\widetilde{X}_0 \subset \widetilde{X}$ and X_0, $\widetilde{X}_0$ closed in $\mathcal{X}$ and Y, $\widetilde{Y}$, Y_0, $\widetilde{Y}_0$ be subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$, $\widetilde{Y}_0 \subset \widetilde{Y}$ and Y_0, $\widetilde{Y}_0$ closed in $\mathcal{Y}$. If $S \in \mathcal{B}(X/X_0, Y/Y_0)$ and $\widetilde{S} \in \mathcal{B}(\widetilde{X}/\widetilde{X}_0, \widetilde{Y}/\widetilde{Y}_0)$ then $\delta(\overline{G_0(\widetilde{S})}, \overline{G_0(S)}) = \delta(G_0(S^*), G_0(\widetilde{S}^*))$.*

Proof. Using Lemmas 3(i), 1(iii) and the fact that J is an isometric isomorphism we deduce that

$$\begin{aligned}\delta\big(G_0(S^*), G_0(\widetilde{S}^*)\big) &= \delta\big(J\big(G_0(S)\big)^{\perp}, J\big(G_0(\widetilde{S})\big)^{\perp}\big) = \delta\big(\overline{J\big(G_0(\widetilde{S})\big)}, \overline{J\big(G_0(S)\big)}\big) \\ &= \delta\big(J\big(\overline{G_0(\widetilde{S})}\big), J\big(\overline{G_0(S)}\big)\big) = \delta\big(\overline{G_0(\widetilde{S})}, \overline{G_0(S)}\big). \qquad \square\end{aligned}$$

3. Proof of the perturbation result for q-morphisms

3.1. Preliminary results

In the following proposition we prove an Albrecht–Vasilescu-type result for q-morphisms acting between normed spaces.

Proposition 1. *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X, $\widetilde{X}$, X_0, $\widetilde{X}_0$ be subspaces of $\mathcal{X}$ such that $X_0 \subset X$, $\widetilde{X}_0 \subset \widetilde{X}$ and X_0, $\widetilde{X}_0$ closed in $\mathcal{X}$ and Y, $\widetilde{Y}$, Y_0, $\widetilde{Y}_0$ be subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$, $\widetilde{Y}_0 \subset \widetilde{Y}$ and Y_0, $\widetilde{Y}_0$ closed in $\mathcal{Y}$. Assume that $S \in \mathcal{B}(X/X_0, Y/Y_0)$ is open,*

$$\dim\big(N(S)\big) < \infty \quad \textit{and} \quad \operatorname{codim}\big(\overline{R(S)}\big) < \infty.$$

There exists $\epsilon_S > 0$ such that if $\widetilde{S} \in \mathcal{B}(\widetilde{X}/\widetilde{X}_0, \widetilde{Y}/\widetilde{Y}_0)$ satisfies

$$\hat{\delta}\big(\overline{G_0(S)}, \overline{G_0(\widetilde{S})}\big) < \epsilon_S, \qquad \hat{\delta}(X_0, \widetilde{X}_0) < \epsilon_S \quad \textit{and} \quad \hat{\delta}(\overline{Y}, \overline{\widetilde{Y}}) < \epsilon_S,$$

then

$$\dim\big(N(\widetilde{S})\big) \leqslant \dim\big(N(S)\big), \qquad \operatorname{codim}\big(\overline{R(\widetilde{S})}\big) \leqslant \operatorname{codim}\big(\overline{R(S)}\big)$$

and

$$\overline{\operatorname{ind}}(\widetilde{S}) \leqslant \overline{\operatorname{ind}}(S).$$

Moreover, $\overline{\operatorname{ind}}(\widetilde{S}) = \overline{\operatorname{ind}}(S)$ iff $\widetilde{S}$ is open.

Proof. Consider the bounded linear operators between Banach spaces given in Remark 2

$$S^* : Y_0^\perp / Y^\perp \to X_0^\perp / X^\perp, \qquad \widetilde{S}^* : \widetilde{Y}_0^\perp / \widetilde{Y}^\perp \to \widetilde{X}_0^\perp / \widetilde{X}^\perp.$$

Because S is open, from Lemma 3(ii) and (iii), we deduce that $R(S^*)$ is closed in $X_0^\perp / X^\perp$ and

$$\dim\big(N(S)\big) = \operatorname{codim}\big(R(S^*)\big), \qquad \operatorname{codim}\big(\overline{R(S)}\big) = \dim\big(N(S^*)\big).$$

Hence,

$$S^* \text{ is Fredholm} \quad \text{and} \quad \operatorname{ind}(S^*) = -\overline{\operatorname{ind}}(S).$$

Applying Theorem 3 to S^* and $\widetilde{S}^*$ we obtain that there exists $\epsilon_S > 0$ such that if

$$\widehat{\delta}\big(X_0^\perp, \widetilde{X}_0^\perp\big) < \epsilon_S, \qquad \widehat{\delta}\big(Y^\perp, \widetilde{Y}^\perp\big) < \epsilon_S \quad \text{and} \quad \widehat{\delta}\big(G_0(S^*), G_0(\widetilde{S}^*)\big) < \epsilon_S,$$

then $\widetilde{S}^*$ is Fredholm,

$$\dim\big(N(\widetilde{S}^*)\big) \leqslant \dim\big(N(S^*)\big), \qquad \operatorname{codim}\big(R(\widetilde{S}^*)\big) \leqslant \operatorname{codim}\big(R(S^*)\big)$$

and

$$\operatorname{ind}(\widetilde{S}^*) = \operatorname{ind}(S^*).$$

In particular we have that $R(\widetilde{S}^*)$ is closed in $\widetilde{X}_0^\perp / \widetilde{X}^\perp$ and using again Lemma 3(ii), it follows that

$$\operatorname{codim}\big(\overline{R(\widetilde{S})}\big) = \dim\big(N(\widetilde{S}^*)\big) \leqslant \dim\big(N(S^*)\big) = \operatorname{codim}\big(\overline{R(S)}\big),$$
$$\dim\big(N(\widetilde{S})\big) \leqslant \operatorname{codim}\big(R(\widetilde{S}^*)\big) \leqslant \operatorname{codim}\big(R(S^*)\big) = \dim\big(N(S)\big)$$

and

$$\overline{\operatorname{ind}}(\widetilde{S}) \leqslant -\operatorname{ind}(\widetilde{S}^*) = -\operatorname{ind}(S^*) = \overline{\operatorname{ind}}(S).$$

Moreover, using that $R(\widetilde{S}^*)$ is closed in $\widetilde{X}_0^\perp / \widetilde{X}^\perp$ and $\operatorname{codim}(R(\widetilde{S}^*)) < \infty$, it follows from Lemma 3(iii) that $\dim(N(\widetilde{S})) = \operatorname{codim}(R(\widetilde{S}^*))$ iff $\widetilde{S}$ is open. Hence, $\widetilde{S}$ is open iff $\overline{\operatorname{ind}}(\widetilde{S}) = \overline{\operatorname{ind}}(S)$.

If we also note that (see Lemmas 1(iii) and 4)

$$\widehat{\delta}\big(X_0^\perp, \widetilde{X}_0^\perp\big) = \widehat{\delta}(X_0, \widetilde{X}_0), \qquad \widehat{\delta}\big(Y^\perp, \widetilde{Y}^\perp\big) = \widehat{\delta}(\overline{\widetilde{Y}}, \overline{Y})$$

and

$$\widehat{\delta}\big(G_0(S^*), G_0(\widetilde{S}^*)\big) = \widehat{\delta}\big(\overline{G_0(S)}, \overline{G_0(\widetilde{S})}\big)$$

the conclusion follows. □

Remark 3. (i) Let X, Y be two normed spaces and $T : X \to Y$ be a linear, bounded operator with closed range. Results (3.3.7.1), (3.5.6.1), (6.12.1.2) and (6.12.1.21) from the monograph [13] imply that the following properties are equivalent.

(a) There exists $\rho > 0$ such that $\rho B_Y \cap R(T) \subset T(B_X)$ (that is T is open in our settings), $\dim(N(T)) < \infty$ and $\operatorname{codim}(R(T)) < \infty$.
(b) $\dim(N(T')) < \infty$ and $\operatorname{codim}(R(T')) < \infty$ (that is T' is Fredholm in our settings).

In this case we have that $\overline{\operatorname{ind}}(T) = -\operatorname{ind}(T')$.

(ii) Under the assumptions of Proposition 1, if we suppose in addition that $R(\widetilde{S})$ is closed in $\widetilde{Y}/\widetilde{Y}_0$, then using that $\widetilde{S}'$ is Fredholm, it follows from (i) that $\widetilde{S}$ is open. Hence, in this case one has that $\overline{\operatorname{ind}}(\widetilde{S}) = \overline{\operatorname{ind}}(S)$.

Lemma 5. *Let X, Y be normed spaces X_0 be a closed subspace of X and Y_0 be a closed subspace of Y.*

(i) *The operator $S : X \to Y/Y_0$ is open iff the operator*

$$S_0 : G_0(S) \to Y, \quad S_0(x, y) = y$$

is open.

(ii) *Assume that $S_0 : X \to Y$ is open, $S_0(X_0) \subset Y_0$ and $Y_0 \subset R(S_0)$. Then*

$$S : X/X_0 \to Y/Y_0, \quad S(x + X_0) = S_0(x) + Y_0$$

(*called the q-morphism induced by S_0*) *is open.*

(iii) *Let $S_0 \in \mathcal{B}(X, Y)$ be such that $S_0(X_0) = Y_0$. Assume that the q-morphism $S : X/X_0 \to Y/Y_0$ induced by S_0 is open and $S_0|X_0 : X_0 \to Y_0$ is open. Then S_0 is open.*

Proof. (i) Assume that S is open and consider $k > 0$ such that for any $y + Y_0 \in R(S)$, there exists $x \in X$ with

$$Sx = y + Y_0 \quad \text{and} \quad \|x\| \leqslant k\|y + Y_0\| \leqslant k\|y\|.$$

This implies, in particular, that for every $y \in R(S_0)$, there exists $(x, y) \in G_0(S)$,

$$y = S_0(x, y) \quad \text{and} \quad \big\|(x, y)\big\| \leqslant (k + 1)\|y\|.$$

Conversely, assume that S_0 is open and consider $\rho > 0$ such that $\rho B_Y \cap R(S_0) \subset S_0(B_{G_0(S)})$. Take $0 < \rho' < \rho$ and $y + Y_0 \in \rho' B_{Y/Y_0} \cap R(S)$. There exists $y_0 \in Y_0$ such that $\|y + y_0\| < \rho$ and $y + y_0 \in R(S_0)$. Using the openness of S_0 we deduce that there exists $x \in X$ such that $\|(x, y + y_0)\| \leqslant 1$ and $(x, y + y_0) \in G_0(S)$. In particular we have that $\|x\| \leqslant 1$ and $S(x) = y + Y_0$. Hence $\rho' B_{Y/Y_0} \cap R(S) \subset S(B_X)$, so S is open.

(ii) Since S_0 is open we can find $\rho > 0$ such that $\rho B_Y \cap R(S_0) \subset S_0(B_X)$. Take $0 < \rho' < \rho$ and $y + Y_0 \in R(S)$ such that $\|y + Y_0\| \leqslant \rho'$. Then there exists $y_1 \in y + Y_0$ such that $\|y_1\| \leqslant \rho$. Using $y + Y_0 \in R(S)$ it follows that there exists $x \in X$ such that $S(x + X_0) = y + Y_0$, so $y - S_0(x) \in Y_0 \subset R(S_0)$. It follows that $y \in R(S_0)$ and using again that $Y_0 \subset R(S_0)$, we deduce that $y_1 \in R(S_0)$. Hence, there exists $x_1 \in B_X$ such that $S_0(x_1) = y_1$. Then, $S(x_1 + X_0) = S_0(x_1) + Y_0 = y + Y_0$ and $\|x_1 + X_0\| \leqslant \|x_1\| \leqslant 1$. Consequently $\rho' B_{Y/Y_0} \cap R(S) \subset S(B_{X/X_0})$, that is S open.

(iii) Because S is open there is $k > 0$ such that for all $y + Y_0 \in R(S)$, there exists $x + X_0 \in X/X_0$ satisfying $S(x + X_0) = y + Y_0$ and $\|x + X_0\| \leqslant k\|y + Y_0\|$. If we take $k' > k$ we can find $x' \in X$ such that $S(x' + X_0) = S_0(x') + Y_0 = y + Y_0$ and $\|x'\| \leqslant k'\|y + Y_0\|$. Taking into account that $R(S) = R(S_0)/Y_0$ we obtain that for all $y \in R(S_0)$ there exists $x' \in X$ and $y_0 \in Y_0$ such that $y = S_0(x') + y_0$ and $\|x'\| \leqslant k'\|y\|$. Because $S_0|X_0$ is open and $S_0(X_0) = Y_0$, there is $k'' > 0$ such that, for every $y_0 \in Y_0$ there exists $x'' \in X_0$ such that $S_0(x'') = y_0$ and $\|x''\| \leqslant k''\|y_0\|$. Let $y \in R(S_0)$. As we have noticed before, we can find $x' \in X$ and $x'' \in X_0$ such that

$$y = S_0(x' + x''), \qquad \|x'\| \leqslant k'\|y\| \quad \text{and} \quad \|x''\| \leqslant k''\big\|S_0(x'')\big\|.$$

Therefore, if we take $x = x' + x''$ we have that $S_0(x) = y$ and

$$\begin{aligned}\|x\| &\leqslant \|x'\| + \|x''\| \leqslant k'\|y\| + k''\big\|y - S_0(x')\big\| \leqslant k'\|y\| + k''\|y\| + k''\|S_0\|\|x'\| \\ &\leqslant \big(k' + k'' + k'k''\|S_0\|\big)\|y\|.\end{aligned}$$

Consequently, S_0 is open. □

The idea of the following lemma comes from [6].

Lemma 6. *Let $\mathcal{U}, \mathcal{V}$ be normed spaces U_0, U be subspaces of $\mathcal{U}$ such that the closure of U_0 in $\mathcal{U}$ is a subset of U and V_0, V be subspaces of $\mathcal{V}$ such that $V_0 \subset V$ and V_0 is closed in $\mathcal{V}$. Let $L : U/U_0 \to V/V_0$ be a q-morphism such that there exists $L_0 \in \mathcal{B}(U, V)$ satisfying*

$$L_0(U_0) \subset V_0, \quad L(u + U_0) = L_0 u + V_0.$$

Then

$$P : U/\overline{U}_0 \to V/V_0, \quad P(u + \overline{U}_0) = L_0 u + V_0$$

is a bounded operator. Moreover $R(P) = R(L)$ and $N(P) = L_0^{-1}(V_0)/\overline{U}_0$. In particular,

$$\dim\big(N(L)\big) < \infty \quad \textit{and} \quad \operatorname{codim}\big(\overline{R(L)}\big) < \infty$$

iff

$$\dim\big(N(P)\big) < \infty, \qquad \operatorname{codim}\big(\overline{R(P)}\big) < \infty \quad \textit{and} \quad \dim(\overline{U}_0/U_0) < \infty.$$

In this case

$$\dim\big(N(L)\big) = \dim\big(N(P)\big) + \dim(\overline{U}_0/U_0),$$
$$\operatorname{codim}\big(\overline{R(L)}\big) = \operatorname{codim}\big(\overline{R(P)}\big),$$
$$\overline{\operatorname{ind}}(L) = \overline{\operatorname{ind}}(P) + \dim(\overline{U}_0/U_0).$$

Proof. Let $u_1, u_2 \in U$ be such that $u_1 - u_2 \in \overline{U}_0$. Using $L_0(\overline{U}_0) \subset V_0$ it follows that $L_0(u_1 - u_2) \in V_0$, so the operator P is well defined. On the other hand, we have that

$$\big\|P(u_1 + \overline{U}_0)\big\| = \big\|L_0(u_2) + V_0\big\| \leqslant \big\|L_0(u_2)\big\| \leqslant \|L_0\|\|u_2\|.$$

As u_2 was chosen arbitrarily in $u_1 + \overline{U}_0$, it follows that

$$\big\|P(u_1 + \overline{U}_0)\big\| \leqslant \|L_0\|\|u_1 + \overline{U}_0\|,$$

so P is bounded. It is clear that

$$R(P) = R(L), \qquad N(P) = L_0^{-1}(V_0)/\overline{U}_0$$

and, because

$$N(P) \simeq \big(L_0^{-1}(V_0)/U_0\big)/(\overline{U}_0/U_0) = N(L)/(\overline{U}_0/U_0),$$

we obtain the desired conclusion. □

The following result has been proved in the Banach space context in [7, Lemma I.9.4].

Proposition 2. *Let X, Y be two linear spaces, $X_0 \subset X$, $Y_0 \subset Y$ be linear subspaces and $T : X/X_0 \to Y/Y_0$ be a q-morphism. Consider the q-morphisms*

$$S_1 : G_0(T)/(X_0 \times Y_0) \to X/X_0, \quad S_1\big((x, y) + X_0 \times Y_0\big) = x + X_0$$

and

$$S_2 : G_0(T)/(X_0 \times Y_0) \to Y/Y_0, \quad S_2\big((x, y) + X_0 \times Y_0\big) = y + Y_0.$$

Then, S_1 is bijective and $T = S_2 S_1^{-1}$. In particular

$$\dim\big(N(T)\big) = \dim\big(N(S_2)\big), \qquad R(T) = R(S_2).$$

Proof. It is clear that S_1 and S_2 are well defined and a simple calculation shows that S_1 is bijective and $S_1^{-1}(x + X_0) = (x, y) + X_0 \times Y_0$, where $y \in Y$ satisfies $T(x + X_0) = y + Y_0$. Fix $x \in X$. There exists $y \in Y$ such that $(x, y) \in G_0(T)$. We have that

$$\big(S_2 S_1^{-1}\big)(x + X_0) = S_2\big((x, y) + X_0 \times Y_0\big) = y + Y_0 = T(x + X_0),$$

from where we obtain that $T = S_2 S_1^{-1}$. It follows that $R(T) = R(S_2)$ and $N(T) \simeq N(S_2)$. This completes the proof. □

3.2. Proof of Theorem 1

Consider the linear operators

$$S_0 : G_0(T) \to Y, \quad S_0(x, y) = y \quad \text{and} \quad \widetilde{S}_0 : G_0(\widetilde{T}) \to Y, \quad \widetilde{S}_0(x, y) = y.$$

We have that

$$\big\|S_0(x, y)\big\| = \|y\| \leqslant \big\|(x, y)\big\|, \quad \forall (x, y) \in G_0(T).$$

It follows that $S_0 \in \mathcal{B}(G_0(T), Y)$ and analogously one has that $\widetilde{S}_0 \in \mathcal{B}(G_0(\widetilde{T}), Y)$. Using Proposition 2 we associate to T and to $\widetilde{T}$ respectively, the q-morphisms

$$S_2 : G_0(T)/(X_0 \times Y_0) \to Y/Y_0, \quad S_2\big((x, y) + X_0 \times Y_0\big) = S_0(x, y) + Y_0,$$
$$\widetilde{S}_2 : G_0(\widetilde{T})/(X_0 \times \widetilde{Y}_0) \to \widetilde{Y}/\widetilde{Y}_0, \quad S_2\big((x, y) + X_0 \times \widetilde{Y}_0\big) = \widetilde{S}_0(x, y) + \widetilde{Y}_0,$$

and using Lemma 6, the fact that $Y_0, \widetilde{Y}_0$ are closed in $\mathcal{Y}$, $\overline{X}_0 \times Y_0 \subset G_0(T)$ and $\overline{X}_0 \times \widetilde{Y}_0 \subset G_0(\widetilde{T})$, we associate to S_2 and to $\widetilde{S}_2$ respectively, the bounded operators

$$S : G_0(T)/(\overline{X}_0 \times Y_0) \to Y/Y_0, \quad S\big((x, y) + \overline{X}_0 \times Y_0\big) = S_0(x, y) + Y_0,$$
$$\widetilde{S} : G_0(\widetilde{T})/(\overline{X}_0 \times \widetilde{Y}_0) \to \widetilde{Y}/\widetilde{Y}_0, \quad \widetilde{S}\big((x, y) + \overline{X}_0 \times \widetilde{Y}_0\big) = \widetilde{S}_0(x, y) + \widetilde{Y}_0.$$

Note also that $G_0(T) = G_0(\phi_T)$, hence from the openness of ϕ_T and Lemma 5(i) it follows that S_0 is open. This together with $S_0(\overline{X}_0 \times Y_0) = Y_0$ and Lemma 5(ii) imply that S is open. On the other hand, using Proposition 2 and Lemma 6 we deduce that

$$\begin{aligned} &\dim(\overline{X}_0/X_0) < \infty, \\ &\dim\big(N(T)\big) = \dim\big(N(S)\big) + \dim(\overline{X}_0/X_0), \\ &\operatorname{codim}\big(\overline{R(T)}\big) = \operatorname{codim}\big(\overline{R(S)}\big) \end{aligned} \tag{1}$$

and

$$\overline{\operatorname{ind}}(T) = \overline{\operatorname{ind}}(S) + \dim(\overline{X}_0/X_0). \tag{2}$$

We are now in position to apply Proposition 1 to S and $\widetilde{S}$. We obtain the existence of a positive constant $\epsilon_S > 0$ such that, if

$$\widehat{\delta}\big(\overline{G_0(S)}, \overline{G_0(\widetilde{S})}\big) < \epsilon_S, \qquad \widehat{\delta}(\overline{Y}, \overline{\widetilde{Y}}) < \epsilon_S \quad \text{and} \quad \widehat{\delta}(\overline{X}_0 \times Y_0, \overline{X}_0 \times \widetilde{Y}_0) < \epsilon_S,$$

then

$$\dim\big(N(\widetilde{S})\big) \leqslant \dim\big(N(S)\big), \qquad \operatorname{codim}\big(\overline{R(\widetilde{S})}\big) \leqslant \operatorname{codim}\big(\overline{R(S)}\big) \tag{3}$$

and

$$\overline{\operatorname{ind}}(\widetilde{S}) \leqslant \overline{\operatorname{ind}}(S). \tag{4}$$

Moreover, $\overline{\operatorname{ind}}(\widetilde{S}) = \overline{\operatorname{ind}}(S)$ if and only if $\widetilde{S}$ is open.

Using again Proposition 2 and Lemma 6 it follows

$$\begin{aligned} &\dim\big(N(\widetilde{T})\big) = \dim\big(N(\widetilde{S})\big) + \dim(\overline{X}_0/X_0), \\ &\operatorname{codim}\big(\overline{R(\widetilde{T})}\big) = \operatorname{codim}\big(\overline{R(\widetilde{S})}\big) \end{aligned} \tag{5}$$

and

$$\overline{\operatorname{ind}}(\widetilde{T}) = \overline{\operatorname{ind}}(\widetilde{S}) + \dim(\overline{X}_0/X_0). \tag{6}$$

From (1)–(6) we obtain that

$$\dim\big(N(\widetilde{T})\big) \leqslant \dim\big(N(T)\big), \qquad \operatorname{codim}\big(\overline{R(\widetilde{T})}\big) \leqslant \operatorname{codim}\big(\overline{R(T)}\big)$$

and

$$\overline{\operatorname{ind}}(\widetilde{T}) \leqslant \overline{\operatorname{ind}}(T),$$

with equality if and only if $\widetilde{S}$ is open. Note that $R(\widetilde{S}_0|(\overline{X}_0 \times \widetilde{Y}_0)) = \widetilde{Y}_0$ and $\widetilde{S}_0|(\overline{X}_0 \times \widetilde{Y}_0)$ is open (because for every $y \in \widetilde{Y}_0$ one has that $\widetilde{S}_0(0, y_0) = y_0$ and $\|(0, y_0)\| = \|y_0\|$). It follows that

$$\widetilde{S} \text{ is open} \quad \Longleftrightarrow \quad \widetilde{T} \text{ is q-open}.$$

Indeed, if $\widetilde{S}$ is open, using Lemma 5(iii) it follows that $\widetilde{S}_0$ is open and using Lemma 5(i) and $G_0(\widetilde{T}) = G_0(\phi_{\widetilde{T}})$, it follows that $\widetilde{T}$ is q-open. Reciprocally, if $\widetilde{T}$ is q-open, then, using Lemma 5(i) we obtain that $\widetilde{S}_0$ is open and using Lemma 5(ii) we deduce that $\widetilde{S}$ is open. Hence,

$$\overline{\operatorname{ind}}(\widetilde{T}) = \overline{\operatorname{ind}}(T) \quad \Longleftrightarrow \quad \widetilde{T} \text{ is q-open}.$$

Note also that

$$\widehat{\delta}(\overline{X}_0 \times Y_0, \overline{X}_0 \times \widetilde{Y}_0) \leqslant \widehat{\delta}(Y_0, \widetilde{Y}_0). \tag{7}$$

Indeed, let $(x, y) \in \overline{X}_0 \times Y_0$ be such that $\|x\| + \|y\| \leqslant 1$ and $z \in \widetilde{Y}_0$. We have that

$$\inf_{(x_1, z_1) \in \overline{X}_0 \times \widetilde{Y}_0} \big\|(x, y) - (x_1, z_1)\big\| \leqslant \big\|(x, y) - (x, z)\big\| = \|y - z\|.$$

This implies that

$$\inf_{(x_1, z_1) \in \overline{X}_0 \times \widetilde{Y}_0} \big\|(x, y) - (x_1, z_1)\big\| \leqslant \inf_{z_1 \in \widetilde{Y}_0} \|y - z_1\|$$

and

$$\delta(\overline{X}_0 \times Y_0, \overline{X}_0 \times \widetilde{Y}_0) \leqslant \delta(Y_0, \widetilde{Y}_0),$$

and interchanging Y_0 and $\widetilde{Y}_0$ in the above inequality it follows that

$$\delta(\overline{X}_0 \times \widetilde{Y}_0, \overline{X}_0 \times Y_0) \leqslant \delta(\widetilde{Y}_0, Y_0).$$

Now, (7) follows from the above estimates. If we prove that

$$\widehat{\delta}\big(\overline{G_0(S)}, \overline{G_0(\widetilde{S})}\big) \leqslant 2\widehat{\delta}\big(\overline{G_0(T)}, \overline{G_0(\widetilde{T})}\big) + \widehat{\delta}(Y_0, \widetilde{Y}_0) \tag{8}$$

the proof will be completed. We have that

$$G_0(S) = \big\{(x, y, y + y_0) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Y};\ (x, y) \in G_0(T),\ \ y_0 \in Y_0\big\},$$
$$G_0(\widetilde{S}) = \big\{\big(x_1, y_1, y_1 + y^0\big) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Y};\ (x_1, y_1) \in G_0(\widetilde{T}),\ \ y^0 \in \widetilde{Y}_0\big\},$$

which together with Y_0 and $\widetilde{Y}_0$ closed in $\mathcal{Y}$ imply that

$$\overline{G_0(S)} = \big\{(x, y, y + y_0) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Y};\ (x, y) \in \overline{G_0(T)},\ \ y_0 \in Y_0\big\},$$
$$\overline{G_0(\widetilde{S})} = \big\{\big(x_1, y_1, y_1 + y^0\big) \in \mathcal{X} \times \mathcal{Y} \times \mathcal{Y};\ (x_1, y_1) \in \overline{G_0(\widetilde{T})},\ \ y^0 \in \widetilde{Y}_0\big\}.$$

Let $(x, y, y + y_0) \in \overline{G_0(S)}$ be such that $\|x\| + \|y\| + \|y + y_0\| \leqslant 1$. It follows that $(x, y) \in \overline{G_0(T)}$, $y_0 \in Y_0$ and $\|y_0\| = \|y + y_0 - y\| \leqslant \|y\| + \|y + y_0\| \leqslant 1$. For any $(x_1, y_1) \in \overline{G_0(\widetilde{T})}$ and $y^0 \in \widetilde{Y}_0$ we have that $(x_1, y_1, y_1 + y^0) \in \overline{G_0(\widetilde{S})}$ and

$$\begin{aligned}\big\|(x, y, y + y_0) - \big(x_1, y_1, y_1 + y^0\big)\big\| &= \|x - x_1\| + \|y - y_1\| + \big\|(y - y_1) + \big(y_0 - y^0\big)\big\| \\ &\leqslant 2\big(\|x - x_1\| + \|y - y_1\|\big) + \big\|y_0 - y^0\big\|.\end{aligned}$$

This shows that

$$\delta\big(\overline{G_0(S)}, \overline{G_0(\widetilde{S})}\big) \leqslant 2\delta\big(\overline{G_0(T)}, \overline{G_0(\widetilde{T})}\big) + \delta(Y_0, \widetilde{Y}_0),$$

which together with the corresponding inequality obtained interchanging S and $\widetilde{S}$ imply (8).

Remark 4. Under the assumptions of Theorem 1, if we suppose in addition that $R(\widetilde{T})$ is closed $\widetilde{Y}/\widetilde{Y}_0$, then from $R(\widetilde{T}) = R(\widetilde{S})$ (see Lemma 6 and the preceding proof) and Remark 3 it follows that $\widetilde{S}$ is open, so $\widetilde{T}$ is open (see the proof above). Consequently, in this case one has that $\overline{\operatorname{ind}}(\widetilde{T}) = \overline{\operatorname{ind}}(T)$.

3.3. Some consequences

The following consequence of Theorem 1 is the key ingredient in the proof of our application to linear relations.

Corollary 1. *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X, $\widetilde{X}$ be subspaces of $\mathcal{X}$ and Y, $\widetilde{Y}$, Y_0, $\widetilde{Y}_0$ be subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$, $\widetilde{Y}_0 \subset \widetilde{Y}$ and Y_0, $\widetilde{Y}_0$ are closed in $\mathcal{Y}$. Let $T : X \to Y/Y_0$ be an open linear operator satisfying*

$$\dim\big(N(T)\big) < \infty \quad \text{and} \quad \operatorname{codim}\big(\overline{R(T)}\big) < \infty.$$

Then there exists $\epsilon_T > 0$ such that for every linear operator $\widetilde{T} : \widetilde{X} \to \widetilde{Y}/\widetilde{Y}_0$ satisfying

$$\widehat{\delta}\big(\overline{G_0(T)}, \overline{G_0(\widetilde{T})}\big) < \epsilon_T, \qquad \widehat{\delta}(Y_0, \widetilde{Y}_0) < \epsilon_T \quad \text{and} \quad \widehat{\delta}(\overline{Y}, \overline{\widetilde{Y}}) < \epsilon_T$$

one has that

$$\dim\big(N(\widetilde{T})\big) \leqslant \dim\big(N(T)\big), \qquad \operatorname{codim}\big(\overline{R(\widetilde{T})}\big) \leqslant \operatorname{codim}\big(\overline{R(T)}\big)$$

and

$$\overline{\operatorname{ind}}(\widetilde{T}) \leqslant \overline{\operatorname{ind}}(T).$$

Moreover, $\overline{\operatorname{ind}}(\widetilde{T}) = \overline{\operatorname{ind}}(T)$ iff $\widetilde{T}$ is open.

Another interesting consequence of Theorem 1 is given in the following

Corollary 2. *Let $\mathcal{X}$, $\mathcal{Y}$ be normed spaces, X, $\widetilde{X}$, X_0 be subspaces of $\mathcal{X}$ such that $X_0 \subset X \cap \widetilde{X}$ and Y, $\widetilde{Y}$, Y_0, $\widetilde{Y}_0$ be subspaces of $\mathcal{Y}$ such that $Y_0 \subset Y$ and $\widetilde{Y}_0 \subset \widetilde{Y}$. Let $T : X/X_0 \to Y/Y_0$ be a q-open q-morphism with $G_0(T)$ closed in $\mathcal{X} \times \mathcal{Y}$,*

$$\dim\big(N(T)\big) < \infty \quad \textit{and} \quad \operatorname{codim}\big(\overline{R(T)}\big) < \infty.$$

Then there exists $\epsilon_T > 0$ such that for every q-morphism $\widetilde{T} : \widetilde{X}/X_0 \to \widetilde{Y}/\widetilde{Y}_0$ with $G_0(\widetilde{T})$ closed in $\mathcal{X} \times \mathcal{Y}$,

$$\widehat{\delta}\big(G_0(T), G_0(\widetilde{T})\big) < \epsilon_T, \qquad \widehat{\delta}(Y_0, \widetilde{Y}_0) < \epsilon_T \quad \textit{and} \quad \widehat{\delta}(\overline{Y}, \overline{\widetilde{Y}}) < \epsilon_T$$

we have that

$$\dim\big(N(\widetilde{T})\big) \leqslant \dim\big(N(T)\big), \qquad \operatorname{codim}\big(\overline{R(\widetilde{T})}\big) \leqslant \operatorname{codim}\big(\overline{R(T)}\big)$$

and

$$\overline{\operatorname{ind}}(\widetilde{T}) \leqslant \overline{\operatorname{ind}}(T).$$

Moreover, $\overline{\operatorname{ind}}(\widetilde{T}) = \overline{\operatorname{ind}}(T)$ iff $\widetilde{T}$ is q-open.

Proof. Using that $G_0(T)$ is closed in $\mathcal{X} \times \mathcal{Y}$ it follows easily that Y_0 is closed in $\mathcal{Y}$ and $\overline{X}_0 \times Y_0 \subset G_0(T)$. Analogously, using that $G_0(\widetilde{T})$ is closed in $\mathcal{X} \times \mathcal{Y}$ it follows easily that $\widetilde{Y}_0$ is closed in $\mathcal{Y}$ and $\overline{X}_0 \times \widetilde{Y}_0 \subset G_0(\widetilde{T})$. Now, Corollary 2 follows from Theorem 1. □

Remark 5. Assume that in the preceding corollary one has that $\widetilde{T} = T + A$ where $A_0 \in \mathcal{B}(X, Y)$, $A_0(X_0) \subset Y_0$ and $A : X/X_0 \to Y/Y_0$ is the q-morphism induced by A_0. Then

$$\widehat{\delta}\big(G_0(T), G_0(\widetilde{T})\big) \leqslant \|A_0\|.$$

For the proof, remark that

$$G_0(T + A) = \big\{(x, y + A_0x);\ (x, y) \in G_0(T)\big\}.$$

We prove first that

$$\delta\big(G_0(T), G_0(T + A)\big) \leqslant \|A_0\|.$$

Fix $(x, y) \in G_0(T)$ such that $\|(x, y)\| \leqslant 1$. Then

$$\inf_{\eta \in G_0(T+A)} \big\|(x, y) - \eta\big\| \leqslant \big\|(x, y) - (x, y + A_0x)\big\| = \|A_0x\| \leqslant \|A_0\|.$$

This implies that

$$\sup_{\substack{\xi \in G_0(T) \\ \|\xi\| \leqslant 1}} \inf_{\eta \in G_0(T+A)} \|\xi - \eta\| \leqslant \|A_0\|,$$

so, $\delta(G_0(T), G_0(T + A)) \leqslant \|A_0\|$. Note that

$$G_0(T) = \big\{(x, y - A_0x);\ (x, y) \in G_0(T + A)\big\}.$$

Similar arguments allow us to deduce that

$$\delta\big(G_0(T + A), G_0(T)\big) \leqslant \|A_0\|$$

and the conclusion follows.

4. Applications to linear relations

Remark 6. Let X, Y be normed spaces and $Z \subset X \times Y$ be a linear relation. In order to prove Theorem 2 we associate to Z a linear operator defined by

$$Q_Z : D(Z) \to Y/M(Z), \quad Q_Z(x) = y + M(Z),$$

whenever $(x, y) \in Z$. The map Q_Z is well defined. Indeed, if (x, y), $(x, y_1) \in Z$, then $(0, y - y_1) \in Z$, so $y - y_1 \in M(Z)$. Hence, $y + M(Z) = y_1 + M(Z)$. Clearly Q_Z is linear, $R(Q_Z) = R(Z)/M(Z)$ and $N(Q_Z) = N(Z)$.

Note that for any linear relation $Z \subset X \times Y$ with $M(Z)$ closed, one has that

$$Z \text{ is open} \iff Q_Z \text{ is open}.$$

Indeed, if Z is open, then using [10, Remark II.3.6] it follows that Q_Z is open. Reciprocally, assume that Q_Z is open. Then, using Lemma 5(i) it follows that the linear operator

$$p_2 : G_0(Q_Z) \to Y, \quad p_2(x, y) = y$$

is open, which together with $R(p_2) = R(Z)$ and $G_0(Q_Z) = Z$ imply that Z is open.

Remark 7. (i) Let X be a normed space and $M \subset N \subset X$ be linear subspaces such that M is closed in X. Then N is closed if and only if N/M is closed.

(ii) Let $Z \subset X \times Y$ be a linear relation with $M(Z)$ closed. Then

$$\operatorname{codim}\bigl(\overline{R(Z)}\bigr) = \operatorname{codim}\bigl(\overline{R(Q_Z)}\bigr).$$

Indeed, we have that $R(Q_Z) = R(Z)/M(Z)$. Using (i) we obtain that $\overline{R(Q_Z)} \subset \overline{R(Z)}/M(Z)$. Conversely, let $y \in \overline{R(Z)}$. There exists $(y_n)_n \subset R(Z)$ such that $y_n \to y$. Because $\pi : Y \to Y/M(Z)$ is continuous, we obtain that $y_n + M(Z) \to y + M(Z)$. Since $(y_n + M(Z))_n \subset R(Q_Z)$ it follows that $y + M(Z) \in \overline{R(Q_Z)}$. The assertion is now established.

Proof of Theorem 2. Consider the q-morphisms $Q_{Z_j} : D(Z_j) \to Y/M(Z_j)$ associated to the linear relations Z_j $(j = 1, 2)$. Note that $G_0(Q_{Z_j}) = Z_j$ $(j = 1, 2)$. On the other hand, using Remarks 6 and 7 it follows that,

$$\dim\bigl(N(Q_{Z_j})\bigr) = \dim\bigl(N(Z_j)\bigr), \qquad \operatorname{codim}\bigl(\overline{R(Q_{Z_j})}\bigr) = \operatorname{codim}\bigl(\overline{R(Z_j)}\bigr)$$

and

$$Z_j \text{ is open} \iff Q_{Z_j} \text{ is open}$$

for $j = 1, 2$. Now the result follows from Corollary 1. □

Remark 8. Under the assumptions of Theorem 2, if we suppose in addition that $R(Z_2)$ is closed, then using that $R(Q_{Z_2}) = R(Z_2)/M(Z_2)$ and Remark 7(i) it follows that $R(Q_{Z_2})$ is closed. Thus, using Remark 4 we deduce that $\overline{\operatorname{ind}}(Z_1) = \overline{\operatorname{ind}}(Z_2)$.

Remark 9. (i) Let $Z \subset X \times Y$. One can consider its *norm* given by

$$\|Z\| = \sup_{\|x\| \leqslant 1} \inf_{(x,y) \in Z} \|y\|.$$

If $\|Z\| < \infty$, then the relation Z is said to be *continuous* (see also [10, Proposition II.3.2]).

(ii) If in the above theorem Z_1 and Z_2 are closed linear relations such that

$$Z_2 = Z_1 + \widetilde{Z}_1,$$

where $\widetilde{Z}_1 \subset X \times Y$ is a continuous linear relation such that $D(Z_1) \subset D(\widetilde{Z}_1)$ and $M(\widetilde{Z}_1) \subset M(Z_1)$, then $M(Z_1)$ is closed and $M(Z_1) = M(Z_2)$, hence $\widehat{\delta}(M(Z_1), M(Z_2)) = 0$. Moreover, we have that

$$\widehat{\delta}(Z_1, Z_2) \leqslant \|\widetilde{Z}_1\|.$$

Indeed, let us fix $(x, y) \in Z_1$ such that $\|(x, y)\| \leqslant 1$. Because $x \in D(Z_1) \subset D(\widetilde{Z}_1)$ it follows that there exists $y_1 \in Y$ such that $(x, y_1) \in \widetilde{Z}_1$. Then

$$\inf_{\eta \in Z_2} \bigl\|(x, y) - \eta\bigr\| \leqslant \bigl\|(x, y) - (x, y + y_1)\bigr\| = \|y_1\|.$$

As y_1 was chosen arbitrary with the property that $(x, y_1) \in \widetilde{Z}_1$ and because $\|x\| \leqslant 1$, it follows that

$$\inf_{\eta \in Z_2} \bigl\|(x, y) - \eta\bigr\| \leqslant \inf_{(x, y_1) \in \widetilde{Z}_1} \|y_1\| \leqslant \|\widetilde{Z}_1\|.$$

Hence, we have that $\delta(Z_1, Z_2) \leqslant \|\widetilde{Z}_1\|$. We show now that $\delta(Z_2, Z_1) \leqslant \|\widetilde{Z}_1\|$. Fix $(x, z) \in Z_2$ with $\|(x, z)\| \leqslant 1$. There exists $y_1, \widetilde{y}_1 \in Y$ such that $z = y_1 + \widetilde{y}_1$, $(x, y_1) \in Z_1$ and $(x, \widetilde{y}_1) \in \widetilde{Z}_1$. Let $y \in Y$ be such that $(x, y) \in \widetilde{Z}_1$. It follows that $\widetilde{y}_1 - y \in M(\widetilde{Z}_1) \subset M(Z_1)$, so $(x, z - y) \in Z_1$. This implies that $\inf_{\xi \in Z_1} \|(x, z) - \xi\| \leqslant \|y\|$ and $\inf_{\xi \in Z_1} \|(x, z) - \xi\| \leqslant \inf_{(x,y) \in \widetilde{Z}_1} \|y\|$, hence, because $\|x\| \leqslant 1$ it follows that $\inf_{\xi \in Z_1} \|(x, z) - \xi\| \leqslant \|\widetilde{Z}_1\|$, and our claim holds.

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